Convective and Rotational Stability of a Dilute Plasma

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Received ______________________; accepted ______________________

ABSTRACT

The stability of a dilute plasma to local convective and rotational disturbances is examined. A subthermal magnetic field and finite thermal conductivity along the field lines are included in the analysis. Stability criteria similar in form to the classical Høiland inequalities are found, but with angular velocity gradients replacing angular momentum gradients, and temperature gradients replacing entropy gradients. These criteria are indifferent to the properties of the magnetic field and to the magnitude of the thermal conductivity. Angular velocity gradients and temperature gradients are both free energy sources; it is not surprising that they are directly relevant to the stability of the gas. Magnetic fields and thermal conductivity provide the means by which these sources can be tapped. Previous studies have generally been based upon the classical Høiland criteria, which are inappropriate for magnetized, dilute astrophysical plasmas. In sharp contrast to recent claims in the literature, the new stability criteria demonstrate that marginal flow stability is not a fundamental property of accreting plasmas thought to be associated with low luminosity X-ray sources.

Subject headings: accretion, accretion disks—black hole physics—convection—hydrodynamics—instabilities—turbulence

1. Introduction

Accretion onto compact objects is generally possible only if specific angular momentum is somehow extracted from fluid elements. It is now known that magnetic fields in a differentially rotating fluid cause a breakdown of laminar flow into turbulence (Balbus & Hawley 1998), and that this turbulence leads to a much enhanced angular momentum transport. Despite the fact that turbulence is to some extent amenable to detailed numerical simulation, the theory of turbulent transport lacks a good phenomenological description. It may well be that one does not exist.

To make progress, some form of idealization is usually necessary, and a common approach has been the following: the magnetohydrodynamic (MHD) turbulent transport is modeled as an enhanced Navier-Stokes viscosity in an *unmagnetized* fluid. The idea is that the flow of this ansatz hydrodynamical fluid may then be subject to further instabilities. In particular, the enhanced viscosity is also an energy source, and heating in the inner regions of some types of accretion flow leads, it is argued, to a convectively-unstable temperature gradient. It has been further argued that, much like stellar convective zones, the flow would quickly evolve to a state of marginal stability. Proponents of these convection-dominated accretion flows ('CDAFs') suggest that this process explains the very low X-ray luminosities associated with many black-hole candidates (Narayan, Igumenschev, & Abramowicz 2000), because the *inward* angular momentum transport associated with thermal convection would precisely offset the usual outward MHD transport. In the stellar case, marginal stability amounts to adopting an adiabatic temperature profile; in an accretion flow matters are more complex, with constant entropy and angular momentum surfaces satisfying marginal stability by the Høiland criterion (Quataert & Gruzinov 2000).

We are therefore motivated to consider the equilibrium of a rotating, magnetized, hot dilute plasma. The results of this investigation may be surprising to the reader, because they are at odds with what has become a standard approach. Our findings are relevant to understanding current problems in X-ray accretion sources, but they are also of general fluid dynamical interest. They are easily stated: (1) In a rotating, stratified, dilute plasma, the presence of any magnetic field and any Coulomb-based thermal conductivity renders the classical Høiland criteria insufficient for flow stability (to axisymmetric disturbances). Maximum growth rates are rapid (dynamical time scale), and insensitive to the field strength and the value of the conductivity. (2) Where entropy (S) and angular momentum (l) gradients appear in the classical Høiland formulae, they must be replaced with temperature (T) and angular velocity (Ω) gradients. *Spherical adiabatic accretion is unstable.* (3) Convective instability is inseparable from rotational instability, in the sense that the breakdown of laminar flow into turbulent eddies and the formation of convective eddies are inseparable processes, governed by identical intertwined criteria. Marginal stability is no more likely to be attained in advection-dominated type flows than it is in disks. Indeed, it is the departure from marginal stability that is critical for maintaining vigorous turbulent transport, transport which is at once affected by the dynamical and the thermal properties of the flow.

The following section presents a detailed development of these results, and the final section of this paper discusses some the consequences for understanding the stability of hot plasma winds and accretion flows.

2. Local Stability of a Dilute Plasma

2.1. Physical Description

The appearance of Ω and T gradients as stability discriminants has fundamental significance. They strongly destabilize classically stable flows. Before entering a technical discussion, it is desirable to have a physical understanding of how this arises in some illustrative cases.

In figure (1), we contrast the stability behavior of rotating fluid elements in an unmagnetized (1a) and magnetized (1b) medium. In the unmagnetized case, the fluid element retains its angular momentum l_1 on its epicyclic excursion. The angular momentum of an undisturbed orbit at the new location is l_2. The Rayleigh criterion for stability is simply $l_1 < l_2$, so that the displaced element would drop back to a lower orbit. This means that the angular momentum in the disk increases outward. The magnetized medium is distinguished by magnetically tethered fluid elements, which enforces isorotation at marginal stability, i.e., $\Omega_1 = \Omega_2$. Now the criterion for stability (angular momentum less than surroundings) is $\Omega_1 < \Omega_2$. Hence, a magnetized disk requires that the angular velocity, not the angular momentum, increase outward for stability. This is the magnetorotational stability criterion.

Figure (2) shows the configuration appropriate to a thermally stratified medium. In the unmagnetized case (2a), the fluid element retains its entropy S_1 as it is displaced. The entropy of the undisturbed layer at the element's new location is S_2. The Schwarzschild criterion for stability is $S_1 < S_2$, which is just the condition for the blob to be denser than its (pressure-equilibrium) surroundings. This means the entropy increases upwards.

In the magnetized case (2b), the displaced fluid element draws magnetic field lines up with it, and the temperature gradient now has a component along the displaced field line. Heat flows along this direction from the hotter to the cooler element. Under conditions of marginal stability, thermal conduction maintains a constant temperature between the fluid elements. The criterion for buoyant stability is now $T_1 < T_2$. A magnetized medium requires that the temperature, not the angular momentum, increase upward for stability. We shall now show that the classical Høiland criteria are simply but significantly modified to accommodate these new stability criteria.

2.2. Analysis

In the presence of a magnetic field, heat is restricted to flowing along lines of magnetic force, provided that the ion Larmor radius is much less than the collisional mean free path (e.g. Braginskii 1965). This is equivalent to

$$\omega_{cI}\tau \gg 1 \tag{1}$$

where ω_{cI} is the ion cyclotron frequency, and τ is the mean free collision time. This inequality is generally satisfied by dilute astrophysical plasmas. Under these circumstances, the electron heat conduction parallel to the magnetic field is given by

$$\boldsymbol{Q} = -\chi\boldsymbol{b}\,(\boldsymbol{b}\cdot\boldsymbol{\nabla})T_e \tag{2}$$

where χ is the electron conductivity (Spitzer 1962),

$$\chi \simeq 6 \times 10^{-7}\, T_e^{5/2} \text{ ergs cm}^{-1}\text{ K}^{-1}\,, \tag{3}$$

and T_e is the electron temperature. $\boldsymbol{b}$ is a unit vector in the direction of the magnetic field. Unless otherwise explicitly stated, we shall assume that the ions and electrons have the same temperature T. This amounts to requiring that the mean free path λ and the flow scale height H satisfy $\lambda/H \ll 1$ (Cowie and McKee 1977), or

$$10^4 \frac{T^2}{nH} \ll 1. \tag{4}$$

The usual fluid equations for a monotomic plasma are

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla\cdot}(\rho \boldsymbol{v}) = 0, \tag{5}$$

$$\rho \frac{\partial \boldsymbol{v}}{\partial t} + (\rho \boldsymbol{v\cdot\nabla})\boldsymbol{v} = -\boldsymbol{\nabla}\left(P + \frac{B^2}{8\pi}\right) - \rho \boldsymbol{\nabla}\Phi + \left(\frac{\boldsymbol{B}}{4\pi}\boldsymbol{\cdot\nabla}\right)\boldsymbol{B}, \tag{6}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{v} \times \boldsymbol{B}), \tag{7}$$

$$\frac{3}{2}P\frac{d\ln P\rho^{-5/3}}{dt} = -\boldsymbol{\nabla\cdot Q} = \boldsymbol{\nabla\cdot}\left[\boldsymbol{b}\,(\chi \boldsymbol{b\cdot\nabla}T)\right]. \tag{8}$$

These are respectively mass conservation, the equation of motion, the induction equation, and the entropy equation. Our notation is standard: $\boldsymbol{v}$ is the fluid velocity, ρ the mass density, P the gas pressure, $\boldsymbol{B}$ the magnetic field, and Φ the gravitational potential. The ion-dominated viscosity is ignored, as it is smaller than the conductivity by a factor of order the square root of the electron-to-ion mass ratio. (This assumes equal ion and electron temperatures.) The plasma is taken to be a perfect electrical conductor.

We assume that in the equilibrium solution, the field lines are isotherms. Then the linearly perturbed heat flux is

$$\boldsymbol{\delta Q} = -\chi \boldsymbol{b}\,(\boldsymbol{\delta b}\cdot\boldsymbol{\nabla})T - \chi \boldsymbol{b}(\boldsymbol{b}\cdot\boldsymbol{\nabla})\delta T. \tag{9}$$

Perturbed quantities have the WKB space-time dependence $\exp i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)$, where $\boldsymbol{k}$ is the axisymmetric wavevector $(k_R, 0, k_Z)$. We shall work in the Boussinesq limit, and the linearly perturbed equations are therefore

$$\boldsymbol{\nabla\cdot\delta v} = 0, \tag{10}$$

$$\frac{\partial \delta \boldsymbol{v}}{\partial t} + \delta \boldsymbol{v\cdot\nabla v} = \frac{\delta\rho}{\rho^2}\boldsymbol{\nabla}P - \frac{1}{\rho}\boldsymbol{\nabla}\left(\delta P + \frac{\boldsymbol{\delta B}\cdot\boldsymbol{B}}{4\pi}\right) + \frac{(\boldsymbol{B}\cdot\nabla)}{4\pi\rho}\boldsymbol{\delta B}, \tag{11}$$

$$\frac{\partial \boldsymbol{\delta B}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{\delta v} \times \boldsymbol{B}) + \boldsymbol{\nabla} \times (\boldsymbol{v} \times \boldsymbol{\delta B}), \tag{12}$$

$$\frac{5}{3}\frac{\partial}{\partial t}\frac{\delta\rho}{\rho} - \boldsymbol{\delta v}\cdot\boldsymbol{\nabla}\ln P\rho^{-5/3} = \frac{2}{3P}\boldsymbol{\nabla}\cdot\boldsymbol{\delta Q}. \tag{13}$$

In explicit component form, the leading order WKB equations are

$$k_R\delta v_R + k_Z\delta v_Z = 0, \tag{14}$$

$$\begin{aligned} -i\omega\delta v_R \quad &+\frac{ik_R}{\rho}\delta P - 2\Omega\delta v_\phi - \frac{\delta\rho}{\rho^2}\frac{\partial P}{\partial R} + \frac{ik_R}{4\pi\rho} \\ &\times(B_\phi\,\delta B_\phi + B_Z\,\delta B_Z) - \frac{ik_Z}{4\pi\rho}B_Z\,\delta B_R = 0, \end{aligned} \tag{15}$$

$$-i\omega\delta v_\phi + \delta v_R\,\frac{1}{R}\frac{\partial(R^2\Omega)}{\partial R} + \delta v_Z\,R\frac{\partial\Omega}{\partial Z} - i\boldsymbol{k}\cdot\boldsymbol{B}\frac{\delta B_\phi}{4\pi\rho} = 0, \tag{16}$$

$$\begin{aligned} -i\omega\delta v_Z \quad &+\frac{ik_Z\,\delta P}{\rho} - \frac{\delta\rho}{\rho^2}\frac{\partial P}{\partial Z} + \frac{ik_Z}{4\pi\rho} \\ &\times(B_\phi\,\delta B_\phi + B_R\,\delta B_R) - \frac{ik_R B_R}{4\pi\rho}\,\delta B_Z = 0, \end{aligned} \tag{17}$$

$$-i\omega\delta B_R - i\boldsymbol{k}\cdot\boldsymbol{B}\delta v_R = 0, \tag{18}$$

$$-i\omega\delta B_\phi - \delta B_R\,\frac{\partial\Omega}{\partial\ln R} - \delta B_Z\,R\frac{\partial\Omega}{\partial Z} - i\boldsymbol{k}\cdot\boldsymbol{B}\,\delta v_\phi = 0, \tag{19}$$

$$-i\omega\delta B_Z - i\boldsymbol{k}\cdot\boldsymbol{B}\delta v_Z = 0, \tag{20}$$

$$i\omega\frac{5}{3}\frac{\delta\rho}{\rho} + (\boldsymbol{\delta v}\cdot\boldsymbol{\nabla})\ln P\rho^{-5/3} = \frac{2\chi}{3P}\left(i\boldsymbol{k}\cdot\boldsymbol{b}\,(\boldsymbol{\delta b}\cdot\boldsymbol{\nabla})T - (\boldsymbol{k}\cdot\boldsymbol{b})^2\delta T\right). \tag{21}$$

Note that the change in the unit vector $\boldsymbol{b}$ is

$$\boldsymbol{\delta b} = \boldsymbol{b}\times\left(\frac{\boldsymbol{\delta B}}{B}\times\boldsymbol{b}\right). \tag{22}$$

These equations differ from the *adiabatic* magnetized Høiland criteria studied by Balbus (1995, hereafter B95) only in the final energy equation (21), which contains a rather complicated-looking conduction term. Despite the apparent complexity, the fact that only the energy equation has changed allows us to simplify considerably our calculation of the dispersion relation, by making use of the B95 result.

Begin by setting $\delta T = -T(\delta\rho/\rho)$ in the conduction term, which is an implementation of the Boussinesq approximation (relative changes in the pressure are much smaller than relative changes in the temperature or density). Then, substituting for $\boldsymbol{\delta b}$ from equation (22), and remembering that the field lines are isothermal, brings us to

$$\left(i\omega - \frac{2}{5}\frac{\chi T}{P}(\boldsymbol{k}\cdot\boldsymbol{b})^2\right)\frac{\delta\rho}{\rho} = -\frac{3}{5}\boldsymbol{\delta v}\cdot\boldsymbol{\nabla}\ln P\rho^{-5/3} + \frac{2i\chi T}{5P}(\boldsymbol{k}\cdot\boldsymbol{b})\frac{\boldsymbol{\delta B}}{B}\cdot\boldsymbol{\nabla}\ln T. \tag{23}$$

Using equations (18), (20) and (14), then simplifying, leads to

$$\frac{\delta\rho}{\rho} = \frac{\delta v_R}{i\omega}\left[\frac{\frac{3}{5}\mathcal{D}\ln P\rho^{-5/3} + \frac{2\chi T i}{5P\omega}(\boldsymbol{k\cdot b})^2\mathcal{D}\ln T}{1+\frac{2\chi T i}{5P\omega}(\boldsymbol{k\cdot b})^2}\right] \equiv \frac{\delta v_R}{i\omega}\frac{3\Theta}{5}\mathcal{D}\ln P\rho^{-5/3}, \tag{24}$$

where

$$\mathcal{D} = \left(\frac{k_R}{k_Z}\frac{\partial}{\partial Z} - \frac{\partial}{\partial R}\right), \tag{25}$$

and Θ is defined *in situ.*

Equation (24) has two important limiting forms. When $\chi \to 0$ and $\Theta \to 1$, we recover the adiabatic expression

$$\frac{\delta\rho}{\rho} = \frac{\delta v_R}{i\omega}\,\frac{3}{5}\mathcal{D}\ln P\rho^{-5/3} = -\xi_R\,\frac{3}{5}\mathcal{D}\ln P\rho^{-5/3}, \tag{26}$$

where we have introduced the radial Lagrangian fluid displacement, $\delta v_R = d\xi_R/dt$. This returns the calculation to B95. But note the result obtained if we first take the limit $\omega \to 0$, followed by $\chi \to 0$. Then Θ is proportional to the ratio of the temperature-to-entropy $\mathcal{D}$ gradients, and

$$\frac{\delta\rho}{\rho} = -\xi_R\,\mathcal{D}\ln T. \tag{27}$$

This implies that the $\omega \to 0$ limit of the dispersion relation is simply obtained by taking the B95 result and replacing $(3/5)\mathcal{D}\ln P\rho^{-5/3}$ with $\mathcal{D}\ln T$. Since the $\omega \to 0$ limit is relevant for flow stability, comparison of equations (26) and (27) points to something remarkable: the stability of the flow changes discontinuously when any finite thermal conductivity is present. Instead of entropy gradients serving as the discriminant for buoyant stability, temperature gradients are key. If the temperature — *not* the entropy — decreases upwards, the plasma will become buoyantly unstable. This is a very big difference indeed, implying that simple adiabatic stratification (or flow) is unstable (Balbus 2000). It is wholly analogous to the replacement of angular momentum gradients with angular velocity gradients in a magnetized rotator, and the accompanying destabilization of Keplerian flow.

2.3. Dispersion Relation

The above reasoning suggests a simple way to obtain the desired dispersion relation: use Θ as a "tag," and substitute $3\Theta/5$ for $3/5$ where it appears as a prefactor in equation (2.4) in B95. This prescription gives directly

$$\tilde{\omega}^4\,\frac{k^2}{k_Z^2} + \tilde{\omega}^2\left[\frac{3\Theta}{5\rho}\,(\mathcal{D}P)\,\mathcal{D}\ln P\rho^{-5/3} + \frac{1}{R^3}\,\mathcal{D}(R^4\Omega^2)\right] - 4\Omega^2(\boldsymbol{k}\cdot\boldsymbol{v_A})^2 = 0,$$

where

$$\boldsymbol{v_A} = \boldsymbol{B}/\sqrt{4\pi\rho}, \qquad \tilde{\omega}^2 = \omega^2 - (\boldsymbol{k}\cdot\boldsymbol{v_A})^2, \qquad k^2 = k_R^2 + k_Z^2. \tag{28}$$

We may now expand this equation using (24), introducing $\sigma = -i\omega$ to keep the dispersion coefficients real. One finds:

$$\frac{k^2}{k_Z^2}\tilde{\sigma}^4\sigma_{cond} - \tilde{\sigma}^2\Big[\frac{3}{5\rho}\mathcal{D}P\Big(\sigma\mathcal{D}\ln P\rho^{-5/3} + \frac{2\chi T}{3P}(\boldsymbol{k\cdot b})^2\mathcal{D}\ln T\Big) + \\ + \frac{1}{R^3}\mathcal{D}(R^4\Omega^2)\sigma_{cond}\Big] - 4\Omega^2(\boldsymbol{k\cdot v_A})^2\sigma_{cond} = 0, \quad (29)$$

where now

$$\tilde{\sigma}^2 = \sigma^2 + (\boldsymbol{k\cdot v_A})^2, \quad \sigma_{cond} = \sigma + \frac{2\chi T}{5P}(\boldsymbol{k\cdot b})^2 \quad (30)$$

This is the general form of the dispersion relation.

The marginal stability of purely evanescent modes may be studied very simply by passing through the point $\sigma \to 0$. The condition for stability is

$$(\boldsymbol{k\cdot v_A})^2\frac{k^2}{k_Z^2} - \frac{1}{\rho}(\mathcal{D}P)(\mathcal{D}\ln T) - \frac{1}{R^3}\mathcal{D}(R^4\Omega^2) - 4\Omega^2 > 0. \quad (31)$$

Note that this condition is independent of the conductivity χ, for any finite value of this parameter. By way of contrast, when $\chi = 0$, the $\sigma \to 0$ result is

$$(\boldsymbol{k\cdot v_A})^2\frac{k^2}{k_Z^2} - \frac{3}{5\rho}(\mathcal{D}P)(\mathcal{D}\ln P\rho^{-5/3}) - \frac{1}{R^3}\mathcal{D}(R^4\Omega^2) - 4\Omega^2 > 0, \quad (32)$$

which is the same as the previous condition, except for the replacement of a logarithmic temperature gradient by an entropy gradient.

The condition (31) is actually much more general than the above simple derivation would suggest. It is in fact a necessary and sufficient condition to preclude both instability and overstability. This is established in detail in the Appendix by use of the Routh-Hurwitz criterion.

At this point of the calculation, the route becomes identical to B95, and the stability criteria are obtained by direct substitution of $\mathcal{D}\ln T$ for $(3/5)\mathcal{D}\ln P\rho^{-5/3}$ in equations (2.9) and (2.11) of that paper. For ease of reference and cross comparison, we give three forms of the Høiland stability criteria, and the conditions under which they are valid:

The textbook case (Tassoul 1978) is adiabatic and unmagnetized.

CLASSICAL HØILAND CRITERIA.

$$-\frac{3}{5\rho}(\boldsymbol{\nabla}P)\boldsymbol{\cdot\nabla}\ln P\rho^{-5/3} + \frac{1}{R^3}\frac{\partial R^4\Omega^2}{\partial R} \geq 0, \quad (33)$$

$$\left(-\frac{\partial P}{\partial Z}\right)\left(\frac{\partial R^4\Omega^2}{\partial R}\frac{\partial \ln P\rho^{-5/3}}{\partial Z} - \frac{\partial R^4\Omega^2}{\partial Z}\frac{\partial \ln P\rho^{-5/3}}{\partial R}\right) \geq 0. \quad (34)$$

The B95 result allows for the presence of a weak magnetic field, but ignores thermal conduction.

ADIABATIC, MAGNETIZED CRITERIA.

$$-\frac{3}{5\rho}(\boldsymbol{\nabla} P)\boldsymbol{\cdot}\boldsymbol{\nabla}\ln P\rho^{-5/3}+\frac{\partial\Omega^2}{\partial\ln R}\geq 0, \tag{35}$$

$$\left(-\frac{\partial P}{\partial Z}\right)\left(\frac{\partial\Omega^2}{\partial R}\frac{\partial\ln P\rho^{-5/3}}{\partial Z}-\frac{\partial\Omega^2}{\partial Z}\frac{\partial\ln P\rho^{-5/3}}{\partial R}\right)\geq 0. \tag{36}$$

The result of this paper includes both the dynamics of a weak magnetic field, and the effects of magnetically inhibited Coulomb conductivity.

NONADIABATIC, MAGNETIZED CRITERIA (This paper).

$$-\frac{1}{\rho}(\boldsymbol{\nabla} P)\boldsymbol{\cdot}\boldsymbol{\nabla}\ln T+\frac{\partial\Omega^2}{\partial\ln R}\geq 0, \tag{37}$$

$$\left(-\frac{\partial P}{\partial Z}\right)\left(\frac{\partial\Omega^2}{\partial R}\frac{\partial\ln T}{\partial Z}-\frac{\partial\Omega^2}{\partial Z}\frac{\partial\ln T}{\partial R}\right)\geq 0. \tag{38}$$

The combined effect of a magnetic field and Coulomb conductivity is to ensure that free energy sources (angular velocity and temperature gradients) control flow stability. This result goes some way toward understanding why minimal energy and maximal entropy states of bound systems are associated with uniform rotation and isothermality, yet the classical dynamical stability criteria involve gradients of angular momentum and entropy. Departures from uniform rotation and isothermality are indeed a source of dynamical instability. It is just that magnetic tension and magnetically confined conduction are needed to provide the right coupling to tap into these sources.

2.4. Some additional points

2.4.1. Radiative conduction

Because of the apparent generality of the criteria (37) and (38), it is important to emphasize the point (Balbus 2000) that the destabilizing role of thermal conduction is modified substantially when diffusivity is dominated by radiative processes, as in stellar interiors. This form of the heat conduction is indifferent to the magnetic field. Denoting the radiative conductivity as χ_{rad}, we have (Schwarzschild 1958):

$$\chi_{rad}=\frac{16T^3\sigma}{3\kappa\rho}, \tag{39}$$

where σ is the Stefan-Boltzmann constant and κ is the radiative opacity. When both Coulomb and radiative conductivity are present, the right hand side of equation (21) is modified to

$$\frac{2}{3P}\left[i\chi\boldsymbol{k}\cdot\boldsymbol{b}\,(\boldsymbol{\delta b}\cdot\boldsymbol{\nabla})T-\left(\chi(\boldsymbol{k}\cdot\boldsymbol{b})^2+\chi_{rad}k^2\right)\delta T\right]. \tag{40}$$

(We assume, as before, that unperturbed field lines are isothermal. Our ultimate conclusion is not strongly affected by this assumption.) The dispersion relation becomes

$$\frac{k^2}{k_Z^2}\tilde{\sigma}^4\sigma_{rad} - \tilde{\sigma}^2\Big[\frac{3}{5\rho}\mathcal{D}P\Big(\sigma\mathcal{D}\ln P\rho^{-5/3} + \frac{2\chi T}{3P}(\boldsymbol{k\cdot b})^2\mathcal{D}\ln T\Big) + \\ + \frac{1}{R^3}\mathcal{D}(R^4\Omega^2)\sigma_{rad}\Big] - 4\Omega^2(\boldsymbol{k\cdot v_A})^2\sigma_{rad} = 0, \tag{41}$$

where

$$\sigma_{rad} = \sigma + \frac{2T}{5P}(\chi(\boldsymbol{k\cdot b})^2 + \chi_{rad}k^2). \tag{42}$$

The stability condition (31) becomes

$$(\boldsymbol{k\cdot v_A})^2\frac{k^2}{k_Z^2} - \frac{(1/\rho)(\mathcal{D}P)(\mathcal{D}\ln T)}{1+(\chi_{rad}/\chi)(k/\boldsymbol{k\cdot b})^2} - \frac{1}{R^3}\mathcal{D}(R^4\Omega^2) - 4\Omega^2 > 0. \tag{43}$$

By reducing the effective size of the potentially destabilizing temperature gradient, radiative conductivity is strongly stabilizing. Even when formally present, instability occurs primarily at long wavelengths (along the field lines) when radiative conduction is dominant, and the local WKB approximation we have been using breaks down. When unstable wavelengths are calculated to be in excess of the size of the system, the question of stability must be determined by a global analysis. The dynamical stability of stellar radiative interiors, therefore, is not threatened by this analysis (Balbus 2000).

2.4.2. Instability in a dynamical background

The development of a local instability in an evolving, dynamically active background is more complex than the analogous problem for a static equilibrium (e.g. Balbus & Soker 1989). Both accretion flows and winds are a natural venue for the instabilities of interest, but we shall defer a detailed study of the dynamical linear stability theory to a separate investigation (Balbus 2001, in preparation). It is nevertheless both possible and useful to make some simple statements of a general nature. We restrict our comments to spherical flow.

At $t = 0$, label each fluid element by a position vector $\boldsymbol{r'}$, which is a Lagrangian coordinate system, comoving with the unperturbed flow. The position vector $\boldsymbol{r}$ is the instantaneous Eulerian coordinate of a fluid element as it flows. Radial stretching of the flow is characterized by the scale factor

$$a(t) \equiv \frac{\partial r}{\partial r'},$$

which is a sort of Hubble parameter. The fundamental perturbation variable in a dynamical flow is ξ_r/a, where ξ_r is the radial displacement of a perturbed fluid element relative to the undisturbed flow. Note that we normalize ξ_r relative to a, since the latter tracks the relative separation of two points in the underlying flow. Displacements relative to a track true, physical changes.

If the growth rate of an embedded instability is rapid compared with the background flow evolution, the resulting form for ξ_r/a is rather intuitive:

$$\frac{\xi_r}{a} \propto \exp\left(\int^t \gamma(t')\, dt'\right), \tag{44}$$

where γ is the value of the instantaneous growth rate obtained by ignoring background flow at the fluid element's position at time t'. In a spherical system, it is not difficult to show that the most rapidly growing thermal mode corresponds to

$$\gamma^2 = -\frac{1}{\rho}\frac{\partial P}{\partial r}\frac{\partial \ln T}{\partial r}, \tag{45}$$

which is likely to be of order the sound crossing time. In other words, when the background flow is subsonic, one expects only small quantitative changes in the development of a perturbation from static theory.

The breakdown starts to occur when the flow becomes sonic. Then, there is only one time scale in the problem, and the integral in (44) becomes logarithmic, with power law behavior for the perturbations. Once the flow becomes highly supersonic, there is a complete breakdown of the WKB form (44). Generally, such disturbances are "inflated away," and in any case cannot make contact with upstream fluid. Applications of the dispersion relation or the stability criteria to dynamically active flows should be restricted to their subsonic zones.

3. Discussion

The understanding that temperature and angular velocity gradients regulate flow stability in magnetized dilute plasmas has important consequences. When rotation is unimportant, isothermal, not adiabatic, conditions should prevail. This should be true whether or not field lines are free to open up. Marginal stability is a likely outcome, since the temperature profile is free to adjust itself while maintaining hydrostatic equilibrium. One interesting application is to X-ray gas in early-type galaxies. Fabian et al. (1986) show that the classical Schwarzschild stability criterion implies a minimum mass contained within a confining outer radius r_c of

$$M \geq \frac{5c_0^2 r_0}{2G}\frac{1-(P_c/P_0)^{2/5}}{1-r_0/r_c}, \tag{46}$$

where c_0 is the isothermal sound speed at observed radius r_0, at the pressures P_c and P_0 are those at r_c and r_0. This constraint may be further tightened with the isothermal criterion to

$$M \geq \frac{c_0^2 r_0}{G}\frac{\ln(P_0/P_c)}{1-r_0/r_c}, \tag{47}$$

which also has the added advantage of being rather insensitive to the pressures. (Note that classical cooling flows are not affected by this instability, as the temperature increases outward when radiative losses are important.)

When application is made to a rotating systems, the notion of marginal stability is often inappropriate, for the simple reason that the system may not have the option of changing its rotation law. In Keplerian disks, the instability reduces to the magnetorotational instability, and that is hardly a marginally stable flow. More contemporary examples are ADAFs (Narayan, Mahadevan, & Quataert 1998 for a review), and CDAFs (Narayan et al. 2000), which were mentioned briefly in the Introduction. The latter are particularly striking, as proponents argue that convection itself is capable of suppressing accretion, i.e. that inward convective angular momentum very nearly cancels the outward transport of MHD turbulence. This is a rather bold extrapolation from earlier numerical simulations (Stone & Balbus 1996, Cabot 1996), which suggest an effective α value some three orders of magnitude smaller than that obtained from the magnetorotational instability. Convective turbulence tends to create velocity-temperature correlations that (not surprisingly) do a fine job of transporting heat, and a very poor job of transporting angular momentum. Recent MHD simulations of resistively-heated, non-radiative, quasi-spherical accretion flows (Stone & Pringle 2001) show no evidence of convective stalling.

Part of the difficulty of ascribing a special role to convective angular momentum transport may be grasped from the form of the stability criteria appropriate to these flows, equations (37) and (38). CDAF proponents use, incorrectly, the classical Høiland criteria (33) and (34). Among other problems, this has the effect of masking the dominant magnetorotational instability, which ultimately governs the nature of the angular momentum transport. The formal epicyclic frequency (the term proportional to the angular momentum gradient in [33]) is real-valued in the flows of interest, and gives no hint of rotational instability. Only the adverse entropy gradients, which are clearly associated with convection, would seem to be involved with instability. But this is very misleading.

The appropriate form of the instability criteria (37) and (38) tell a much different story. The point that the thermally driven component is governed by temperature, not entropy, gradients is a relatively minor one in this context. (For this reason, we need not concern ourselves here with the possible complexities of a two-temperature plasma.) Much more important is the conceptual point that convective and rotational instability must be treated on the same footing. There is no separate "viscous" angular momentum transport and "convective" angular momentum transport, any more than there is turbulent transport and magnetic transport. There is a rotationally driven MHD instability that gives rise to a turbulent stress tensor, and there is nothing marginally unstable about the flow. As mentioned above, Stone & Pringle (2001) have performed two-dimensional calculations that bear this out; preliminary three-dimensional simulations are equally supportive (Hawley, Balbus, & Stone 2001, in preparation).

It is, nevertheless, of great interest to understand the MHD *thermal* properties of nonrotating (or uniformly rotating) flows. What are the numerical prospects for studying the thermoclinic instability in nonrotating systems? It is a daunting task, because thermal conduction along magnetic field lines must be isolated, uncontaminated by numerical cross-field diffusion. When the field becomes highly tangled, computations become prohibitive.

Even at this early stage of inquiry, something can be done. The linear instability in a simple vertically stratified box with a weak horizontal field can be demonstrated under Schwarzschild stable conditions (Stone 2001, private communication). It is even possible to see the early stages of nonlinear instability, as shown in figure [3]. If the effect of the instability is generally to comb field lines out, then it may well be possible to make further progress on the numerical front. Either by field line combing or by convection, the tendency

toward isothermality of a bottom-hot dilute stratified plasma seems a likely outcome.

The results of this paper should make very clear the important conceptual point that even the tiniest of magnetic fields can have dramatic consequences for the macroscopic stability of astrophysical plasmas. Classical hydrodynamical results can be qualitatively incorrect, and great care must be taken before uncritically taking them over into magnetized systems. Subtlety need not imply great complexity: the relative simplicity of our fundamental results, equations (37) and (38), is encouragement that the most important MHD stability properties can also be conceptually simple. And since it seems certain that the final word has not been said on the topic, more surprises are likely.

It is a pleasure to thank J. Hawley and C. Terquem for detailed comments on a preliminary draft of this paper which led to significant improvements in the presentation, and M. Fall for pointing out to me that the new stability criteria sharpen dark matter constraints in X-ray clusters. I am also most grateful to J. Stone for providing me with fig. [3], and for discussions related to the numerics of magnetized thermal conduction. This research was supported by NASA grants NAG 5–7500, NAG5-106555, NAG5-9266, and NSF grant AST00-70979.

Appendix: Necessary and Sufficient Criterion for Stability

To prove that the condition (31) is a necessary and sufficient condition for any type of instability, we first expand the polynomial in equation (29) in the form

$$a_0\sigma^5 + a_1\sigma^4 + a_2\sigma^3 + a_3\sigma^2 + a_4\sigma + a_5 = 0. \tag{48}$$

where

$$a_0 = k^2/k_Z^2, \tag{49}$$

$$a_1 = \frac{2\chi T}{5P}\frac{k^2}{k_Z^2}(\boldsymbol{k\cdot b})^2, \tag{50}$$

$$a_2 = 2(\boldsymbol{k\cdot v_A})^2\frac{k^2}{k_Z^2} - \frac{3}{5\rho}\mathcal{D}P\,\mathcal{D}\ln P\rho^{-5/3} - \frac{1}{R^3}\mathcal{D}(R^4\Omega^2), \tag{51}$$

$$a_3 = \frac{2\chi T}{5P}(\boldsymbol{k\cdot b})^2\left[2(\boldsymbol{k\cdot v_A})^2\frac{k^2}{k_Z^2} - \frac{1}{\rho}\mathcal{D}P\,\mathcal{D}\ln T - \frac{1}{R^3}\mathcal{D}(R^4\Omega^2)\right], \tag{52}$$

$$a_4 = (\boldsymbol{k\cdot v_A})^2\left[(\boldsymbol{k\cdot v_A})^2\frac{k^2}{k_Z^2} - \frac{3}{5\rho}\mathcal{D}P\,\mathcal{D}\ln P\rho^{-5/3} - \frac{1}{R^3}\mathcal{D}(R^4\Omega^2) - 4\Omega^2\right], \tag{53}$$

$$a_5 = \frac{2\chi T}{5P}(\boldsymbol{k\cdot b})^2(\boldsymbol{k\cdot v_A})^2\left[(\boldsymbol{k\cdot v_A})^2\frac{k^2}{k_Z^2} - \frac{1}{\rho}\mathcal{D}P\,\mathcal{D}\ln T - \frac{1}{R^3}\mathcal{D}(R^4\Omega^2) - 4\Omega^2\right]. \tag{54}$$

To make things more compact and manageable, notice that

$$\epsilon \equiv -\frac{3}{5\rho}\mathcal{D}P\,\mathcal{D}\ln P\rho^{-5/3} + \frac{1}{\rho}\mathcal{D}P\,\mathcal{D}\ln T = \frac{2}{5\rho P}(\mathcal{D}P)^2 > 0, \tag{55}$$

and if we furthermore define

$$\delta \equiv (k^2/k_Z^2)(\boldsymbol{k}\cdot\boldsymbol{v_A})^2 + 4\Omega^2 > 0, \tag{56}$$

then we may write

$$a_3 = \frac{2\chi T}{5P}(\boldsymbol{k}\cdot\boldsymbol{b})^2(a_2-\epsilon), \quad a_4 = (\boldsymbol{k}\cdot\boldsymbol{v_A})^2(a_2-\delta), \quad a_5 = \frac{2\chi T}{5P}(\boldsymbol{k}\cdot\boldsymbol{b})^2(\boldsymbol{k}\cdot\boldsymbol{v_A})^2(a_2-\delta-\epsilon). \tag{57}$$

For stability, we require that the real part of σ must be less than zero, i.e., that the roots of equation (48) must all lie in the left complex σ plane. Polynomials with this property are known as *Hurwitz* polynomials (e.g. Levinson & Redheffer 1970). If the left side of equation (48) is a Hurwitz polynomial, both pure exponential instability as well as overstability are precluded. This will be so if and only if

$$\begin{vmatrix} a_1 & a_0 & 0 & 0 & 0 \\ a_3 & a_2 & a_1 & a_0 & 0 \\ a_5 & a_4 & a_3 & a_2 & a_1 \\ 0 & 0 & a_5 & a_4 & a_3 \\ 0 & 0 & 0 & 0 & a_5 \end{vmatrix} > 0, \tag{58}$$

and all five determinants obtained by expanding along the diagonal, i.e.,

$$a_1, \qquad \begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix}, \qquad \text{etc.} \tag{59}$$

are each individually > 0 (Levinson & Redheffer 1970). We shall denote the determinant of each $n \times n$ matrix thus obtained as $\det(n)$, so that

$$\det(1) = a_1, \qquad \det(2) = a_1 a_2 - a_0 a_3,$$

and so forth. The requirement that the left side of equation (48) be a Hurwitz polynomial may then be succinctly stated as

$$\det(n) > 0, \qquad n = 1, 2...5. \tag{60}$$

This is the Routh-Hurwitz criterion. The criterion (31) in the text amounts to the condition $a_5 > 0$, which follows directly from the two demands $\det(4) > 0$ and $\det(5) = a_5 \times \det(4) > 0$. We shall now prove that the general Routh-Hurwitz criterion is satisfied for our dispersion relation.

The first non-trivial requirement is

$$\det(2) = a_1 a_2 - a_0 a_3 > 0. \tag{61}$$

The determinant is easily calculated,

$$\det(2) = \left(\frac{k}{k_Z}\right)^2 \frac{2\chi T}{5P} (\boldsymbol{k \cdot b})^2 \epsilon, \tag{62}$$

and is, indeed, positive.

The calculation for det(3) is a bit more complicated,

$$\det(3) = a_3 \det(2) - a_1 \begin{vmatrix} a_1 & a_0 \\ a_5 & a_4 \end{vmatrix}, \tag{63}$$

which simplifies to

$$\det(3) = \left(\frac{k}{k_Z}\right)^2 \left(\frac{2\chi T}{5P}\right)^2 (\boldsymbol{k \cdot b})^4 \epsilon \left[a_2 - \epsilon - (\boldsymbol{k \cdot v_A})^2\right] \tag{64}$$

But the term in square brackets is just

$$(a_2 - \delta - \epsilon) + 4\Omega^2 + (\boldsymbol{k \cdot v_A})^2 \left(\frac{k^2}{k_Z^2} - 1\right). \tag{65}$$

According to equation (57), the sum of the first three grouped terms must be positive if $a_5 > 0$, and the remaining two terms are each manifestly positive. Hence $\det(3) > 0$.

The calculation for det(4) proceeds as follows:

$$\det(4) = a_4 \det(3) - a_5 \begin{vmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_0 \\ a_5 & a_4 & a_2 \end{vmatrix}, \tag{66}$$

The determinant cofactor of a_5 in the above may be expanded and simplified, reducing to

$$\begin{vmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_0 \\ a_5 & a_4 & a_2 \end{vmatrix} = \left(\frac{k}{k_Z}\right)^2 \left(\frac{2\chi T}{5P}\right) (\boldsymbol{k \cdot b})^2 \epsilon \left[a_2 - (\boldsymbol{k \cdot v_A})^2\right] \tag{67}$$

Using this in equation (66) for det(4), after some algebraic excursion we find

$$\det(4) = \left(\frac{k}{k_Z}\right)^2 \left(\frac{2\chi T}{5P}\right)^2 (\boldsymbol{k \cdot b})^4 (\boldsymbol{k \cdot v_A})^2 \epsilon^2 \left[(\boldsymbol{k \cdot v_A})^2 \left(\frac{k^2}{k_Z^2} - 1\right) + 4\Omega^2\right], \tag{68}$$

which is manifestly positive. Finally, as noted, $\det(5) = a_5 \det(4) > 0$, if $a_5 > 0$. We have therefore shown that the condition $a_5 > 0$, equation (31) in the text, ensures that the real parts of the roots of the polynomial in equation (29) all lie in the left half of the complex σ plane, and that the criteria (37) and (38) are necessary and sufficient for the convective-rotational stability of a hot plasma. QED.

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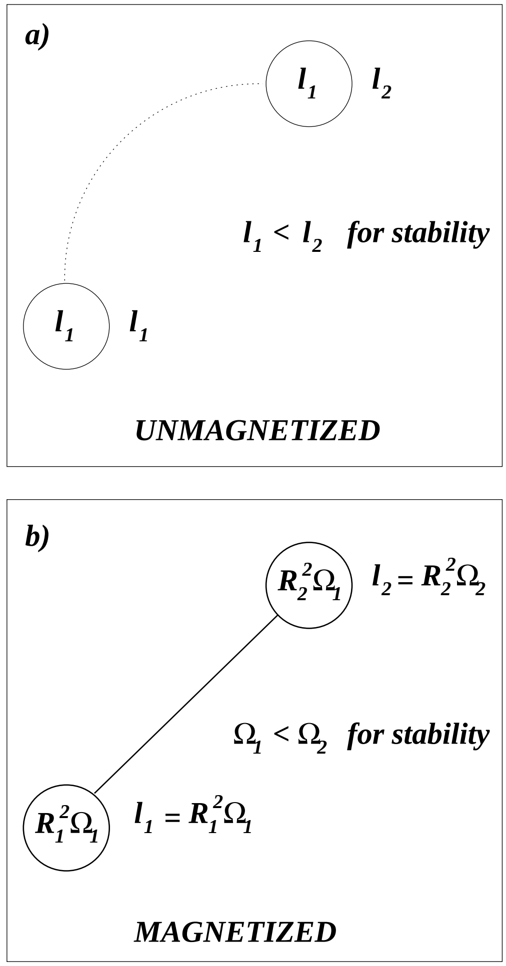

Fig. 1.— (a). In an unmagnetized disk, specific angular momentum is conserved. Stability may be determined by comparing the angular momentum of an outwardly displaced fluid element (l_1) with that of its surroundings (l_2). If $l_1 < l_2$, the displaced fluid element has less angular momentum than it needs to remain in its new orbit, and it drops back. This is equivalent to the Rayleigh condition $dl/dR > 0$. (b). In a magnetized disk, it is the angular velocity of tethered fluid elements that tends to be conserved near marginal stability. Hence, if $\Omega_1 < \Omega_2$, the displaced fluid element has less angular momentum than it needs to remain in its new orbit. This corresponds to the MRI stability condition, $d\Omega/dR > 0$.

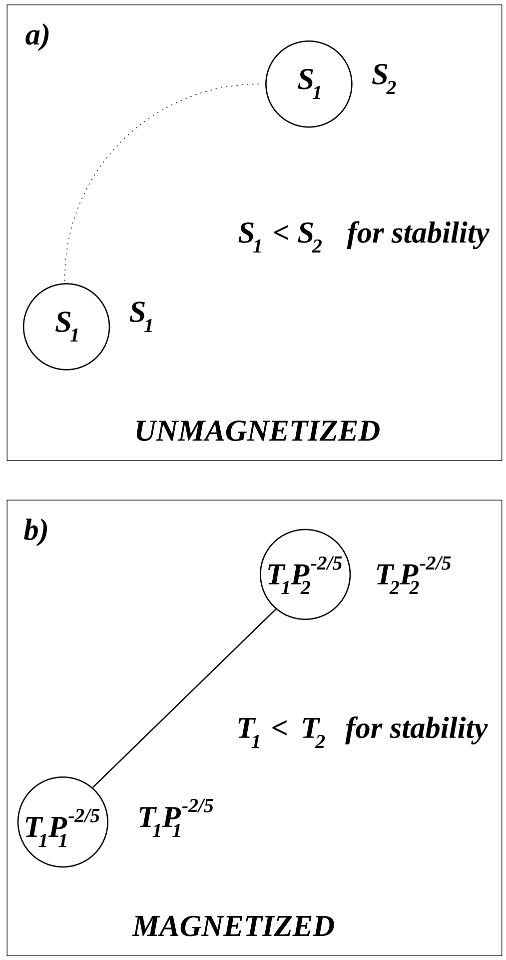

Fig. 2.— A process similar to that depicted in fig. [1] occurs in a thermally stratified disk. (a). In an unmagnetized layer, entropy is conserved. Stability may be determined by comparing the entropy of an outwardly displaced fluid element (S_1) with that of its surroundings (S_2). If $S_1 < S_2$, the displaced fluid element has less entropy than it needs to remain in its new orbit, and it drops back. This is equivalent to the Schwarzschild condition of increasing upward entropy. (b). In a magnetized conducting layer, it is the temperature of tethered fluid elements that tends to be unchanged near marginal stability. As in (a), if $S_1 < S_2$, the displaced fluid element has less entropy than it needs to remain in its new orbit. But since $S \propto \ln TP^{-2/5}$, and pressure balance is maintained, this corresponds to a thermoclinic stability condition: the background temperature must increase upwards.

Fig. 3.— Development of thermoclinic instability in a Schwarzschild-stable layer. Magnetic lines of force are shown after one Alfvén crossing time, initial seeding with rms 1% random initial vertical velocity perturbations. Initial thermal energy density is 1600 times magnetic; initial field lines are isothermal and horizontal; vertical grid runs from $z = 1$ to 2, initial temperature profile is $1/z$, gravitational field is $1/z^2$; χ is 0.05; grid is 128×64. The interpenetration of cool and warm blobs appears very similar to classical convective instability.